

Queen Mary & Westfield College

UNIVERSITY OF LONDON

MAS204 CALCULUS III

18th May 2000 14.30 – 16.30

TIME 2 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators may be used in this examination but any programming, graph plotting or algebraic facility may not be used. Please state on your answer book the name and type of machine used.

You are reminded of the following, which you may use without proof.

In orthogonal curvilinear coordinates (u_1, u_2, u_3) , with corresponding unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and arc-length parameters h_1, h_2, h_3 , the gradient of a scalar field f is given by

$$\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial u_1} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \mathbf{e}_3$$

The divergence of a vector field

$\mathbf{F} = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2 + F_3 \mathbf{e}_3$ is given by

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_3 h_1 F_2) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right],$$

and the curl of the same vector field is given by

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}.$$

In spherical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, \phi)$, the arc-length parameters are $h_1 = 1, h_2 = r, h_3 = r \sin \theta$.

In cylindrical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, z)$, the arc-length parameters are $h_1 = 1, h_2 = r, h_3 = 1$.

SECTION A Each question carries 8 marks. You should attempt ALL questions.

A1. In Cartesian coordinates (x, y, z) , evaluate at the point $(2, 3, 5)$

$$(a) \nabla f, \quad (b) \nabla \cdot \mathbf{F}, \quad (c) \nabla \times \mathbf{F}, \quad (d) \nabla^2 f,$$

where $f = x^3z + y^2$ and $\mathbf{F} = yx^4\mathbf{i} + zy\mathbf{j} + xz^2\mathbf{k}$.

A2. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 2y^2\mathbf{i} + 5z^2\mathbf{j} + 3x^4\mathbf{k}$ and C is a path going from $(0, 0, 0)$ to $(15, 9, 27)$ and is described in parametric form as
 $C: \mathbf{r} = 5t\mathbf{i} + t^2\mathbf{j} + 3t^2\mathbf{k}$

A3. Prove that (a) $\nabla \cdot (\mathbf{r} \times \mathbf{a}) = 0$ and (b) $\nabla \times (\mathbf{r} \times \mathbf{a}) = -2\mathbf{a}$, where $\mathbf{r}$ is the position vector and $\mathbf{a}$ is a constant vector.

A4. For each of the following vector fields $\mathbf{F}$, either find the general solution ϕ satisfying $\mathbf{F} = \nabla\phi$, or show that no such ϕ exists,

$$(a) \mathbf{F} = x^3\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}, \quad (b) \mathbf{F} = z\mathbf{i} + y^2\mathbf{j} + x\mathbf{k}.$$

A5. Show that the following are solutions of Laplace's equation $\nabla^2 f \equiv \nabla \cdot \nabla f = 0$:

$$(a) f = 3\log r + 5r \cos \theta, \quad (b) f = r^2(3 \cos^2 \theta - 1),$$

where (a) is in cylindrical polar coordinates and (b) is in spherical polar coordinates.

A6. Sketch a few representative streamlines for the following velocity fields:

$$(a) \mathbf{u} = y\mathbf{i} + x\mathbf{j}, \quad (b) \mathbf{u} = (\sin y)\mathbf{i}.$$

In each case, determine whether the flow is incompressible ($\nabla \cdot \mathbf{u} = 0$) and whether it is irrotational ($\nabla \times \mathbf{u} = 0$).

A7. Vector fields $\mathbf{B}$ and $\mathbf{A}$ (representing a magnetic field and its vector potential, respectively) are related by $\mathbf{B} = \nabla \times \mathbf{A}$. Suppose that, in cylindrical polar coordinates,

$$\mathbf{A} = B_0 \left(r - \frac{r_0^2}{r} \right) \sin \phi \mathbf{e}_z$$

in the region $r > r_0$ and zero in the region $r < r_0$ (B_0 and r_0 being constants). Evaluate $\mathbf{B}$ in both regions. Is $\mathbf{B}$ continuous across the surface $r = r_0$?

SECTION B Each question carries 22 marks. You may attempt all questions but only marks for the best 2 questions will be counted.

- B1. Consider Laplace's equation $\nabla^2 V = 0$ in the case where V is a function of Cartesian coordinates x and y only. Find all possible separable solutions of the form $V(x, y) = X(x)Y(y)$. Find the function $V(x, y)$ satisfying $\nabla^2 V = 0$ in the region

$$0 < x < 2, \quad 0 < y < 1,$$

subject to the boundary conditions $V = \sin(\pi y)$ along $x = 2$ and $V = 0$ on the other three sides.

- B2. State Stokes' Theorem.

Evaluate

$$\int_S \nabla \times \mathbf{A} \cdot \mathbf{n} dS$$

for

$$\mathbf{A} = y\mathbf{i} + z\mathbf{j} + y\mathbf{k}$$

and the surface S given by

$$z \geq 0, \quad x^2 + y^2 + z^2 = 4$$

- (a) by direct integration

and

- (b) by using Stokes' Theorem.

You may find the following expressions for x , y , z in spherical polar coordinates helpful:

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$

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B3. By using the method of Frobenius, look for the series solution of

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = 0$$

in the form

$$y = \sum_{n \geq 0} a_n x^{n+\sigma}.$$

Show that σ must be equal to 0 or 1, and for each value of σ find a recursion relation for the coefficients a_n .

What is the radius of convergence of each of the two solutions?

In the case of $\sigma = 1$, find explicitly the first four non-zero terms of the power series solution.

B4. A vector field $\mathbf{u}$ with the property that $\mathbf{u} = c \nabla \times \mathbf{u}$, where c is a constant, is called a *Beltrami field*.

- (a) Show that a Beltrami field is solenoidal (i.e. $\nabla \cdot \mathbf{u} = 0$).
- (b) Show that the curl of a Beltrami field is a Beltrami field.
- (c) Suppose that Beltrami field $\mathbf{u}$ has the form $(\sin y, f, g)$ where f and g are functions of Cartesian coordinates x, y, z but g does not depend on x . Find the functions f and g and all possible values of the constant c .

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